RAMAKRISHNA MISSION VIDYAMANDIRA
(Residential Autonomous College under University of Calcutta)

P.G. FIRST YEAR
M.A./M.SC. SECOND SEMESTER (January – June), 2012
Mid-Semester Examination, March 2012

MATHEMATICS

Date : 26/03/2012
Time : 11 am – 3 pm
Full Marks : 100

[Use separate Answer Books for each group]

Group – A

1. Answer any five questions : [5×5]

a) Let F[x] be the polynomial ring over the field F. Assuming that division algorithm holds in F[x] and without proving that it is a Euclidean domain, prove that F[x] is a Principal ideal Domain. Will the result be true if F is just an integral domain? Justify your answer. [3+2]

b) Prove Eisenstien's criterion for irreducibility of a polynomial. Cite one example to show that, a polynomial may be irreducible even when a prime as mentioned in the criterion is not available. [4+1]

c) Prove that any finite extension of a field is an algebraic extension. Give with proper justification an example of an infinite extension which is algebraic. [2+3]

d) Let F be a field of characteristic zero. Let a and b be two algebraic elements in some extension of F. Show that there exists an element c in F(a,b) such that F(a, b)=F (c). Obtain one c such that $Q(\sqrt{3},i)=Q(c)$. [4+1]

e) Let $f(x) = x^4 - 2x^2 - 2 \;\varepsilon\; Q[x]$. Find the roots α and β of f(x) such that Q(α) is isomorphic to Q(β). Find the splitting field of f(x). [3+2]

f) Prove that the product of two primitive polynomials over a UFD is also a primitive polynomial. Hence prove that every irreducible element in a UFD D is a prime in D[x]. [3+2]

g) Define a perfect field. Prove the following :
   i) Any finite field is perfect.
   ii) Any irreducible polynomial over a perfect field is separable. [3+2]

Group – B

2. Attempt all questions : [5×5]

a) Uncountable union of null sets is a null set— prove or disprove.

b) Let X be an infinite set, $\mu = P(x)$, and define $\mu(E) = 0$ if E is finite and $\mu(E) = \infty$ if E is infinite. Then μ is a measure—True or False? Justify your answer.

c) If (x, M, μ) is a measure space and $\{E_j\}_1^\infty \subset M$, then prove that $\mu\left(\underline{\lim}_{j\to\infty} \inf E_j\right) \le \underline{\lim}_{j\to\infty} \inf \mu(E_j)$.

d) Let μ be a premeasure on $\mathcal{A}$, where $\mathcal{A}$ is a field. Define μ* as $\mu^*(E) = \inf\left\{\sum_1^\infty \mu(A_j) : A_j \in \mathcal{A}, E \subset \bigcup_{j=1}^\infty A_j\right\}$

Show that every set in $\mathcal{A}$ is μ* - measurable.
(If μ* is an outermeasure on X, a set $A \subset X$ is called μ*-measurable if
$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^C) \forall\ E \subset X$)

e) If $f, g : X \to \mathbb{R}$ are $(M, B_{\mathbb{R}})$ measurable then max (f,g) and min (f, g) are also $(M, B_{\mathbb{R}})$ measurable.

Group – C

3. Answer any one question : [9×1]

a) State and prove Ascoli – Arzela Theorem.

b) If f(t, x) is continuous on an open set $D \subset R^{n+1}$, then prove that for each point $(\tau, \xi) \in D$, there exists at least one ε-approximate solution of the initial value problem $\dot{x} = f(t, x), x(\tau) = \xi$.

4. Answer <u>any two</u> questions : [6×2]

a) Let f satisfy a Lipschitz condition on $D \subset R^{n+1}$ and let $\varphi_1(t)$ and $\varphi_2(t)$ be two solutions of the initial-value problem $\dot{x} = f(t, x), \xi = x(t)$,

on D. Suppose both $\varphi_1(t)$ and $\varphi_2(t)$ defined on the open interval I. If $\varphi_1(t) = \varphi_2(t)$, then prove that $\varphi_1(t) = \varphi_2(t)$ for every $t \in J$.

b) Define the boundary of a set E.

If x(t) is a maximal solution of $\dot{x} = f(t, x)$ defined on I = {t : a < t < b}, prove that the trajectory (t, x(t)) goes to the boundary of D as t increases to b.

c) Let g(t, y) be continuous and locally Lipschitz on D, an open subset of R^2. Let w(t) be a solution of the differential inequality

$D_\ell w \le g(t, w)$

defined on $I = \{t : a < t \le b\}$, and let y(t) be a solution of $\dot{y} = g(t, y)$ also defined on I. If $w(b) \ge y(b)$, then prove that $w(t) \ge y(t)$ for all $t \in I$.

5. Answer <u>any one</u> question : [4×1]

a) Consider the initial value problem

$\dot{x} = tx,\ x \in R$ and x(0) = 1.

Apply Pi card Method to obtain the solution of the above equation.

b) Let $\mu(t)$ be a nonnegative continuous function of [a,b]. If a continuous function y(t) has the property that

$y(t) \le \lambda + \int_a^t \mu(s) y(s) ds$, $a \le t \le b$, prove that $y(t) \le \lambda e^{\int_a^t \mu(s) ds}$, where λ is a constant.

<u>Group – D</u>

6. Answer <u>any three</u> questions : [25]

a) State the properties of Green's function $G(x, \xi)$ of the following boundary-value problem, $p(x)\frac{d^2u}{dx^2} + q(x)\frac{du}{dx} + r(x)u = f(x),\ \ 0 < x < a$ and $h_1 u(0) + h_2 u'(0) = 0$, $H_1 u(0) + H_2 u'(0) = 0$, Use these properties to show that the solution of the above boundary-value problem is $u(x) = \int_0^a G(x, \xi) f(\xi) d\xi$.

b) Use Green's function to solve the boundary-value problem $\frac{d^2y}{dx^2} + y = x^2,\ 0 < x < \frac{\pi}{2}$; $y(0) = y\left(\frac{\pi}{2}\right) = 0$.

c) Prove the Plemelj's formula, $\lim\limits_{\in \to 0^+} \frac{1}{x + i \in} = -\pi i \delta(x) + P\frac{1}{x}$.

d) i) Show that $\frac{d}{dx}(\operatorname{sgn} x) = 2\delta(x)$.

ii) If $f \in \mathcal{D}'(R^n)$ and $a(x) \in c^\infty(R^n)$, then show that $\frac{\partial}{\partial x_1}(af) = \frac{\partial a_1}{\partial x} f + a\frac{\partial f}{\partial x_1}$.

e) If $f(x) = H[\psi(x)]$, then show that $\psi(x) = -H[f(x)]$, when H is the Hilbert transform operator defined by $H[\varphi(x)] = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\varphi(\xi)}{x - \xi} d\xi$.